Appendix A: Early algebraic thinking: information sheets and consent forms

Early Algebraic Thinking: Links to Numeracy

INFORMATION SHEET FOR STUDENTS, PARENTS and GUARDIANS

Thank you for showing an interest in this project. Please read this information sheet carefully before deciding whether or not to participate. If you decide to participate we thank you. If you decide not to take part there will be no disadvantage to you of any kind and we thank you for considering our request.

1. **What is the aim of the project?**

The aim of the research is to find out about the knowledge and strategic thinking of students as they make the transition from arithmetic to algebra. It is anticipated that eventually this will extend the Number Framework beyond the current highest level. The research questions we want to answer are:

- What knowledge and strategies do students use to solve equations?
- What diagnostic questioning is appropriate for finding out about the knowledge and strategies used by students?

2. **What type of participants are being sought?**

We wish to involve selected students with a range of mathematical ability in Years 7 to 10 of schooling.

3. **What will participants be asked to do?**

Should you agree to take part in this project, you will be asked to:

> Take part in an interview lasting no more than 30 minutes to answer some questions about algebra. A maths teacher from your school will ask the questions, including asking you to explain your thinking. The interview will take place at an agreed time during the school day, but may not be during a regular maths lesson. We hope that what we learn about the way you solve equations will be useful to your maths teacher to support your learning.

Please be aware that you may decide not to take part in the project without any disadvantage to yourself of any kind.

4. **Can participants change their mind and withdraw from the project?**

You may withdraw from participation in the project at any time and without any disadvantage to yourself of any kind.

5. **What data or information will be collected and what use will be made of it?**

Apart from your name (which will be kept confidential), age and year level, the only information we will be collecting will be about the way you solve equations. The interview with you will be recorded on videotape so that the research team can analyse your answers to verbal questions.

This project involves an open-questioning technique where the precise nature of the questions which will be asked have not been determined in advance, but will depend on the way in which the interview develops. Consequently, although the Dunedin College of Education's Research Ethics Committee is aware of the general areas to be explored in the interview, the Ethics Committee has not been able to review the precise questions to be used.

In the event that the line of questioning does develop in such a way that you feel hesitant or uncomfortable, you are reminded of your right to decline to answer any particular question(s) and also that you may withdraw from the project at any stage without any disadvantage to yourself of any kind.

We will use your answers to help us develop a better diagnostic interview, so that all maths teachers will be able to get information on the learning needs of their students.

The only people who will view the videotape of the interview with you or know your name will be the small research team of Dr Chris Linsell, HOD Mathematics at Dunedin College of Education, Noel Johnston and Eric McAuslan, King's High School, Melissa Bell, John McGlashan College, John Bell, Otago Boys' High School and Jan Savell, Numberworks.

Results of this project may be published, but any data included will in no way be linked to any specific participant. You are most welcome to request a copy of the results of the project should you wish.

The data collected will be securely stored in such a way that only those mentioned above will be able to gain access to it. At the end of the project any personal information will be destroyed immediately except that, as required by the College's research policy, any raw data on which the results of a published project depend will be retained in secure storage for five years, after which it will be destroyed.

6. **What if participants have any questions?**

If you have any questions about our project, either now or in the future, please feel free to contact either:

The teacher/researcher from your school or

Dr Chris Linsell
Mathematics Department
Dunedin College of Education
Telephone: 477 2289 ext 753

This project has been reviewed and approved by the Dunedin College of Education's Research Ethics Committee.

Early Algebraic Thinking: Links to Numeracy

CONSENT FORM FOR STUDENTS, PARENTS and GUARDIANS

I have read the Information Sheet concerning this project and understand what it is about. All my questions have been answered to my satisfaction. I understand that I am free to request further information at any stage.

I know that:

1. My participation in the project is entirely voluntary.

2. I am free to withdraw from the project at any time without any disadvantage.

3. The videotapes will be destroyed at the conclusion of the project but if this study is to be published, any raw data on which the results of the project depend will be retained in secure storage for five years, after which it will be destroyed.

4. This project involves an open-questioning technique where the precise nature of the questions which will be asked have not been determined in advance, but will depend on the way in which the interview develops and that in the event that the line of questioning develops in such a way that I (or my child) feel hesitant or uncomfortable I (or may child) may decline to answer any particular question(s) and/or may withdraw from the project without any disadvantage of any kind.

5. The interview will take up to 30 minutes of school time.

6. The results of the project may be published and will be available in the library but every attempt will be made to preserve my anonymity.

I agree to take part in this project.

....

(Signature of participant) (Date)

....

(Signature of parent/guardian) (Date)

This project has been reviewed and approved by the Dunedin College of Education's Research Ethics Committee.

Appendix B: Project timetable

Term 1 2006

- *Literature review.* In order for the teachers to become familiar with the research literature it was decided to give out readings each week, then hold a meeting to discuss what everyone had been reading. Meetings were held with the group of teachers on 2 March, 8 March, 15 March, 22 March, 29 March, 5 April, and 12 April.

  On 8 March, we discussed two chapters on algebra from Linsell's PhD thesis (2005b) as a starting point for looking at the rest of the research literature. It is probably fair to say that the teachers found academic writing rather unfamiliar and difficult.

  On 15 March, we examined the work of Sfard (1991) as process–object duality was identified as being one of the key theoretical frameworks for describing algebraic equations. This was followed up the next week with discussion of the work of Dubinsky and McDonald (2001) and Gray and Tall (1994) in comparison to the ideas of Sfard (1991).

  On 29 March, we discussed students' use of the equals sign, referring to the work of Darr (2003), Kieran (1981), and Saenz-Ludlow and Walgamuth (1998). Students viewing the equals sign as meaning "compute now" rather than "is equivalent to" is seen as one of the important impediments to successful solving of equations.

  At the meeting on 5 April, we looked at some overviews of conceptual development in algebra, using the work of Kieran (1992) and (Thomas & Tall (2001). The classic research by Küchemann (1981) was also discussed.

  At the final meeting of term 1 on 12 April, we looked at research examining other misconceptions held by students about algebra. In particular, Booth (1988) was examined. The "+" sign has an ambiguity, as does the "=" sign. We also discussed MacGregor and Stacey (1997) and looked at:

  – other interpretations of letters in addition to those identified by Küchemann
  – positive effects of teaching
  – interference from new teaching including letter equals 1, exponents, Pythagoras, other symbol systems, and the "fruit salad" approach.

- *Write questions*. It was planned to write questions designed to reveal strategies for solving equations during term 1. Many issues were identified for questions during the process of familiarisation with the research literature and the discussions we had around the literature.

However the process of writing questions did not start until term 2, by which time we had examined most of the important issues in the literature.

Term 2 2006

- *Literature review.* On 13 May, the work of Herscovics and Linchevski (1994) was discussed as being a key paper for informing our diagnostic interview. The comment was made that, after reading the work on process–object duality and conceptual difficulties, this paper now made a lot of sense. Some important points considered were:
  - students' failure to construct meanings and just perform meaningless operations
  - interactions between the hierarchy of mathematical forms of equations and the solution strategies used
  - cognitive demands of operating on unknowns
  - Collis's concept of an operational generalised number
  - use of the = sign
  - order of operations
  - interpretation of symbols, including the difference between $4x$ and $4 \times \square$
  - types and size of numbers in equations
  - the use of all four operations
  - position of unknowns
  - direction of equations
  - number of terms
  - 1 unknown/unknown repeated/unknown on both sides
  - coding of solution processes
  - working around the unknown at the arithmetic level
  - integers.
- *Write questions.* On 13 May, there was considerable discussion to produce a mind map of what might be in our diagnostic interview. For the next meeting, it was planned that we would write some questions for evaluating strategies and would also prepare questions on "knowledge".

  On 27 May, the entire three-hour meeting was spent examining the questions each person had written. The work on strategy questions was based on the questions used by Herscovics

and Linchevski (1994), but did not include all the permutations. We argued that if the knowledge questions assessed knowledge of arithmetic structure and inverse operations, then we did not need all permutations of types of equations in order to ascertain the most advanced strategy a student could use. The solution strategies we were looking for were based on the framework suggested by Kieran (1992):

0. Unable to answer question

1a. Known basic facts

1b. Counting techniques

2. Guess and check

3a. Cover up

3b. Working backwards

4. Formal operations.

It was decided that parallel answer sheets were needed for those students who preferred to work with one option from formal notation, empty boxes or contexts. However, we decided not to make concrete materials available as we found no evidence in the literature or our own experience of them being of assistance to any students. Also, they are not used in the Numeracy Project beyond Advanced Counters. Following the meeting, we produced a diagnostic interview and answer sheet for students.

- *Ethics approval.* This was granted on 3 May.
- *Trial questions with students.* After getting permission slips returned, there were only about 10 days for conducting interviews before our next meeting, but each school managed to interview four students each.
- *View videotapes and discuss effectiveness of questions.* On 24 June each teacher presented the results of the interviews they had conducted during the previous week, showing clips from the videotapes. Some points that stood out were:
  - The importance of doing things and how students were far more capable when they had something concrete or pictorial to work with.
  - We needed to be clearer about the process of exactly how we conducted the interviews.
  - In future we should make copies of the completed questionnaires for everyone to look at during the meeting.
  - We should analyse results question by question rather than by student.
  - We should use the symbol *n* instead of *x* to avoid confusion with the multiplication sign x.

- At this point, we still needed supplementary questions for those questions we thought were knowledge, but might be strategy.

- *Rewrite questions.* More points were raised from the discussion on our first videotapes than expected at the meeting of 24 June, and it was felt there would be benefit in taking more time to analyse the students' responses before revision of the questions. The decision was made to hold the rewriting of questions over until our first meeting of term 3.

Term 3 2006

- *Rewrite questions.* As one of the teachers was a highly experienced Numeracy facilitator, it was decided that we would watch a whole interview of hers on videotape and use this as a model for all to ensure consistency. At the meeting of 29 July, we watched the whole of one of her interviews, stopping it question by question for discussion. We had prepared a summary sheet of the results from all the interviews so that we could discuss what each student had done and modify the questions as required. We decided that the strategy section was the most important so we started by working through that. Following the meeting, we produced a diagnostic interview and answer sheet for students. This was made available for 7 August.

- *Trial questions with students.* One teacher interviewed some Year 7 and Year 8 students. Everyone else interviewed Year 9 and Year 10 students. The aim was for each teacher to interview up to eight students each during the following four weeks. The actual number interviewed was 23 in total. Twenty-two of these interviews were successfully recorded on digital videotape.

- *View videotapes and discuss effectiveness of questions.* All videotapes and transcripts of interviews were collated and summary sheets produced in time for the meeting on 9 September. Because of our previous experiences of the time required for viewing videotapes and modifying questions, we held a full day meeting to do the work. For the first half of the day, we split into two groups of three in order to view the videotapes of strategy interviews and classify the strategies used by the students. For the remainder of the day, we reconvened as a whole group to discuss the students' responses and modify the questions.

- *Rewrite questions.* Using the notes taken during the meeting of 9 September and emailed comments from the teachers on the knowledge questions, we rewrote the questions. Draft 3 of the diagnostic interview was ready for the start of term 4 on 9 October.

Term 4 2006

- *Trial questions with students.* Students from Year 7 to Year 10 were interviewed. The aim was for each of the four teachers and the lead researcher to interview up to eight students each during the following four weeks. The actual number interviewed was 36 in total. All of these interviews were successfully recorded on digital videotape.
- *View videotapes and discuss effectiveness of questions.* All videotapes and transcripts of interviews were collated and summary sheets produced in time for the meeting on 11 November. Again, we held a full day meeting to do the work. For the first half of the day we analysed responses to the strategy questions. For the remainder of the day, we analysed responses to the knowledge questions.
- *Rewrite questions.* Using the notes taken during the meeting of 11 November, we rewrote the questions. Draft 4 of the diagnostic interview was produced (see Appendix C) and is available for use with students in 2007. It should be noted that this is not the physical version of the questionnaire used with students. Strategy questions have been produced on laminated cards so that the interviewer could choose which questions to present to a student and knowledge questions are in the form of a written test.
- *Report on progress to the Otago Mathematics Association*. A workshop was presented to the Otago Mathematics Association mini-conference held on December 1 and 2. Five of the six team members contributed to the presentation, in which we focused on the thinking of students and how we had investigated it.
- *Share findings with Secondary Numeracy Project team*. A full day meeting with Kevin Hannah on November 1 was devoted to discussion of our findings. As a result of the meeting Chris Linsell presented a paper "Solving Equations: Students' Algebraic Thinking" at the National Numeracy Facilitators' Conference in Auckland on 12 to 15 February, 2007.
- *Write research paper on project*. A paper entitled "Using Action Research to Learn About Learning Algebra" was presented at the New Zealand Association for Research in Education (NZARE) Conference in Rotorua on December 5 to 8. This paper has been submitted for publication in the conference proceedings.

Appendix C: Diagnostic interview (draft 4): guidelines for interviewers

This interview should be run in 2 parts (i.e., 2 separate interviews), the knowledge section and the strategy section. Knowledge questions will be done as a written test as supplementary questions are not required. Present questions to groups of students on OHTs if you wish.

Strategy interviews should be recorded on one tape per student for ease of access. To facilitate retrieval of particular parts of interviews, record time of start and end of strategy interview as you make the tapes.

Questions should be presented on cards in different colours so that we can see when next question is given when viewing the tapes. Nametag on each child in front of the camera so that interviews can easily be identified later.

Interviewers should be cautions in the way they make calculators available. Suitable instructions would be: "There is calculator here for when you need to use it. There is pen and paper here for when you need to use it."

Some interviews were hard to hear because of background noise, so the procedure to use is sitting in a quiet room sitting next to the student—signs up to say "no entry—quiet".

Evaluate the strategies used by your students and write these on the sheets. Label videotapes, put protection on videotape after recording and then rewind tape at end.

Codes for strategies are:

0	Attempted, but unable to answer the question
1a	Known basic facts
1b	Counting techniques
1c	Inverse operations
2	Guess and check
3a	Cover up
3b	Working backwards
3c	Working backwards then known facts
3d	Working backwards then guess and check

4 Formal operations

5 Use a diagram

Additional codes:

– Skipped by student

/ Skipped by interviewer

(c) Student used calculator

(e) Student wrote equation

Multiple coding can be used; for example, 5.3b for used diagram to work backwards, 3b.0 for tried unsuccessfully to work backwards or 3d.(c) to indicate worked backwards then used calculator for guess and check

Knowledge section

Subsection A

Question 1 looks at using symbols and letters to represent an unknown that can be found.

1) If $h = 4$ then $5 + h = ?$

Ans = 7

2) If $n = 4$ then $3n = ?$

Ans = 12

Subsection B

Questions 3 to 6 look at manipulating numerals and symbols/unknowns (lack of closure).

3) Add 3 to n

Ans = $n + 3$

4) Add 3 to $d - 1$

Ans = $d + 2$

5) What is 5 times p

Ans = $5p$

6) What is 5 times $y + 2$

Ans = $5(y + 2)$

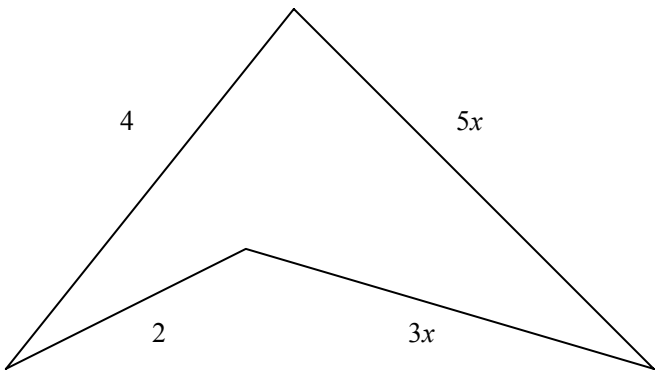
Subsection C

Questions 7 to 12 look at forming expressions with unknowns/symbols in them.

7) Write an equation for the perimeter of the following shape.

Ans = $6 + 8x$

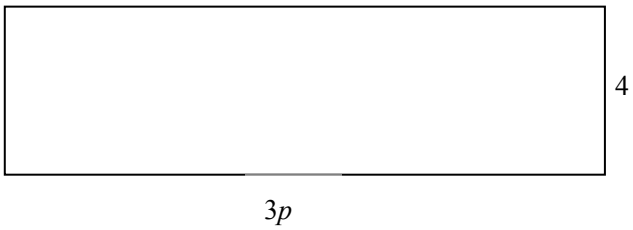
Perimeter =



8) Write an equation for the area of the following shape.

Ans = 12*p*

Area =



Cell phone usage charges are calculated by multiplying the number of seconds you use it by 10 cents, then adding on a cover charge of 55c.

9) If you use the phone for 25 seconds, what will it cost?

Ans = $3.05

10) If you use the phone for *n* seconds, what will it cost?

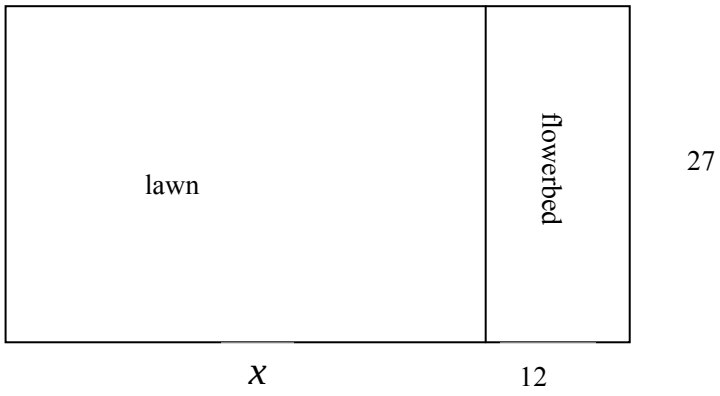
Ans = 10*n* + 55

11) Four times the number of desks in a classroom, minus five, has to equal 103. Which of the following mathematical statements represent this?

Ans = (B) 4D – 5=103

A.	5D – 4 = 103	B.	4D – 5 = 103
C.	D + 4 x 5 = 103	D.	5D – 4 = 103

12) Choose which equation is the correct one to represent the situation.



My garden consists of a lawn and a flowerbed. The total area is 1598 square metres.

Ans = (d) $27(x + 12) = 1598$

a) $27 + 12 + x = 1598$

b) $27 + \frac{x}{12} = 1598$

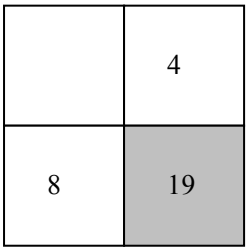
c) $12x + 27 = 1598$

d) $27(x + 12) = 1598$

Subsection D

Questions 13 to 17 look at understanding of the equals sign.

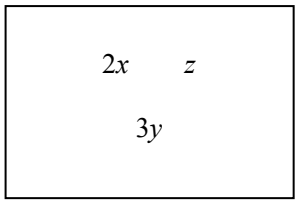
13) By adding the values in the non-shaded squares you get the value in the shaded square. What values should be in the blank square?

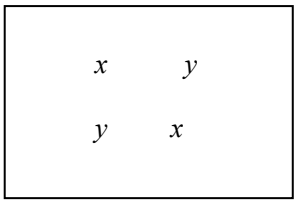


Ans = 7

14) Complete the following pair of diagrams to make the TOTAL in the right box is the same as the total in the left box.

Ans = y + z





Given that $3x + 4 = 15$, find the value of the ? in each of the following

15) $3x + 7 = ?$

Ans = 18

16) $3x + ? = 12$

Ans = 4

17) $5x + 4 = ? + 15$

Ans = $2x$

Subsection E

Questions 18 to 20 look at understanding of number line and operations on integers.

What value of □ makes the following the expressions correct?

18) $^{+}7 + \square = {}^{-}2$

Ans = $^{-}9$

19) $^{+}7 - \square = {}^{+}1$

Ans = 6

20) $^{+}10 - \square = {}^{+}18$

Ans = $^{-}8$

Subsection F

Questions 21 to 24 look at understanding of arithmetic structure.

What answer do you get for each of the following?

21) $5 + 6 \times 10$

Ans = 65

22) $8 \times (7 - 5 + 3)$

Ans = 40

23) $\frac{6 + 12}{3}$

Ans = 6

24) $18 - 12 \div 6$

Ans = 16

Subsection G

Questions 25 to 28 look at understanding of inverse operations.

Replace the boxes with +, –, × or ÷

25) If $63 \div y = 7$ then

$63 = 7 \square y$

Ans is x

26) If $z - 27 = 25$ then

$z = 25 \square 27$

Ans is +

27) If $\frac{p}{6} = 4$ then

$p = 4 \square 6$

Ans is x

28) If $5t = 28$ then

$t = 28 \square 5$

Ans is ÷

Strategy section (abstract)

For the following questions what number could you replace the letter by to make the statement true? This is often called solving equations.

Allow use of calculators.

1) $n - 3 = 12$

($n = 15$)

Purpose: *Determine if able to use known basic facts or counting techniques.*

May also be solved by guess and check.

2) $18 = 3n$

($n = 6$)

Purpose: *Determine if able to use known basic facts, counting techniques, or guess and check when unknown is on the right hand side.*

3) $n + 46 = 113$

($n = 67$)

Purpose: *Determine if able to use working backwards with simple structure.*

4) $\frac{n}{25} = 8$

($n = 200$)

Purpose: *Determine if able to use working backwards when difficulty is increased by operation on unknown.*

5) $4n + 9 = 37$

($n = 7$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack".*

6) $3n - 8 = 19$

($n = 9$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack".*

7) $26 = 10 + 4n$

($n = 4$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack" and unknown is on the right hand side.*

8) $(n + 12) \div 4 = 18$ or $\frac{n + 12}{4} = 18$

($n = 60$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two step process, with division (a more difficult concept or is it?).*

9) $5n + 7 = 15$

($n = 1.6$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack", even when solution is non-integer.*

10) $2 + n \div 4 = 8$

($n = 24$)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two step process, with division and unable to work from left to right.*

11) $5n - 2 = 3n + 6$

($n = 4$)

Purpose: *Determine if able to use formal operations.*

12) $2n - 3 = \frac{2n + 17}{5}$

($n = 4$)

Purpose: *Determine if able to formal operations with REALLY complicated equation!*

13) We can rearrange the equation **p = r – s** to make r the subject, **r = p + s**.

Similarly if v = u + at then a = _____

($a = \frac{v - u}{t}$)

Purpose: *Determine if able to use formal operations with fully symbolic equations involving parameters rather than numbers.*

Strategy section (context)

Allow use of calculators. Provide envelopes of beans, loose beans, and envelopes with the corners chewed off, filled with beans for the bean questions.

14) **I left home this morning with some money, spent $5 and have $17 left. How much did I start with?**

(n = 22)

Purpose: *Determine if able to use known basic facts or counting techniques*

May also be solved by guess and check.

15) **I have 24 CDs. This is three times as many as my brother has. How many CDs does he have?**

(n = 8)

Purpose: *Determine if able to use known basic facts, counting techniques, or guess and check when unknown is on the right hand side.*

16) **If my big sister added my 24 CDs to her collection she would have 89 CDs. How many are in her collection?**

(n = 65)

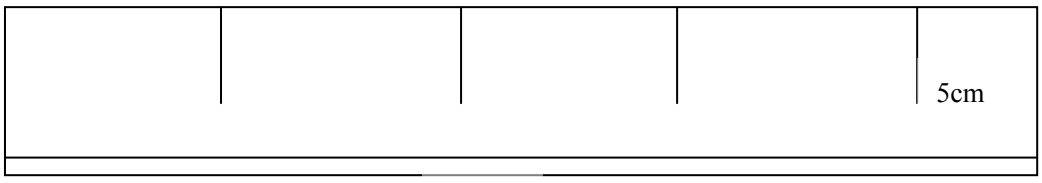
Purpose: *Determine if able to use working backwards with simple structure.*

17) **When I went to bed last night the temperature was 19º and when I woke this morning it was 8º. How much did the temperature increase during the night?**

(n = ¯11)

Purpose: *Determine if able to use working backwards when difficulty is increased by operation on unknown.*

18) **Here are 4 blocks, all the same length, and another block 5cm long. Altogether they are the same length as this big block 37cm long. How long is each of the 4 identical blocks?**



(n = 8)

Purpose: *Determine if able to use diagrams, working backwards, cover up or formal operations when two stages to "unpack".*

19) **I leave home with enough money to buy 3 movie tickets but spend $7 at McDonalds and have now got only $17. How much is the cost of a movie ticket?**

(n = 8)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack".*

20) **On a small bookcase 25cm across, there are 5 books the same thickness and one other book 7cm thick. How thick is each of the identical books?**

(n = 3.6)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two stages to "unpack", even when solution is non-integer.*

21) **Our kapa haka group is made up of some Māori and 11 Pākehā students who are divided into 4 equal groups for practices. Each of the practice groups has 19 students in it. How many Māori students are there in our kapa haka group?**

(n = 65)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two step process, with division (a more difficult concept or is it?).*

22) **Bill's 7m long sailing boat is 2m longer than the length of my house divided by 4. How long is my house?**

(n = 20)

Purpose: *Determine if able to use working backwards, cover up or formal operations when two step process, with division and unable to work from left to right.*

23) **On one shelf of a bookcase I can't quite fit in 5 identical books—the bookcase would need to be 3cm wider. On another shelf of the same length there are 3 of the identical books and a gap of 9cm. How wide is each of the identical books?**

(n = 6)

Purpose: *Determine if able to use formal operations.*

Appendix D: Bibliography of research outputs to date

Linsell, C. (2007, February). *Solving equations: Students' algebraic thinking.* Paper presented at the National Numeracy Conference, Auckland.

Linsell, C., McAusland, E., Bell, M., Savell, J., & Johnston, N. (2006, December). *Using action research to learn about learning algebra.* Paper presented at the annual conference of the New Zealand Association for Research in Education (NZARE), Rotorua.

McAusland, E., Bell, M., Savell, J., Johnston, N., & Linsell, C. (2006, December). *Using action research to learn about learning algebra*. Paper presented at the Otago Mathematics Association Mini Conference, Dunedin.