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Early algebraic thinking: links to numeracy

Many students struggle with introductory algebra and teachers have little to guide them in assisting students to learn this important component of high school mathematics. Little is known about the effect of students' numeracy on the learning of early algebra, or about the strategies that students use to solve equations. There is widespread agreement that algebra is not easily understood by many students. The Cockroft Report in the United Kingdom highlighted the fact that algebra is a source of considerable confusion and negative attitudes among pupils (Cockcroft, 1982), while the title of Brekke's (2001) paper, "School Algebra: Primarily Manipulations of Empty Symbols on a Piece of Paper?", sums up the situation for many students.

The links between numeracy and readiness for learning algebra need to be investigated. Linsell (2005) suggested that only those students who have mastered multiplicative part–whole thinking are capable of solving equations by the formal process of inverse operations. We have available the diagnostic tool for assessing students' stage of numeracy (Ministry of Education, 2003a). To determine how the students' stages of numeracy have an effect on their learning of algebra, a diagnostic tool needs to be developed for assessing algebraic thinking in the domain of solving equations. This would allow a framework for algebraic thinking to start to be developed.

Project aim and objectives

The aim of this research was to make explicit the knowledge and strategic thinking of students as they make the transition from arithmetic to algebra. The development of a diagnostic tool for assessing algebraic thinking in the domain of solving equations was the specific objective for this project. It was therefore proposed that a diagnostic interview, similar to that used in numeracy assessment, be developed. This diagnostic interview will be used in our research on algebraic thinking in 2007 and beyond.

Research questions

The research questions for the project were:

- What knowledge and strategies do students use to solve equations?
- What diagnostic questioning is appropriate for eliciting the knowledge and strategies used by students?

Research methodology

This project took place over one year and used an iterative action-research model to develop the diagnostic interview. A team of teachers and researchers met regularly throughout the year. This team consisted of the lead researcher from the Dunedin College of Education (now the University of Otago's College of Education due to the recent merger), three mathematics teachers from local high schools, one home-room teacher from a Years 7 to 13 college, and a researcher based at the franchise holder of Numberworks, Dunedin. The teachers' practical knowledge of schools and students ensured that the diagnostic interview we developed was appropriate for use with Years 7 to 10 students in schools.

Initially, the teachers needed to become familiar with the research literature on the learning of algebra, in particular the conceptual obstacles (Booth, 1988), process–object duality (Sfard, 1991), and the transition from arithmetic to algebraic thinking (Thomas & Tall, 2001).

The team was then in a position to conjecture the various strategies that students might use, and to make significant contributions to the development of the diagnostic interview questions. We wrote questions designed to reveal these strategies.

Each teacher then trialled these questions with students, recording the interviews on videotape. The teachers had access to students in their own classes plus other Years 7 to Year 10 students in their own schools

We had regular meetings of the whole research group to view the videotapes and discuss the effectiveness of the questions and the students' strategies that were revealed. Some questions were ineffective or redundant, while alternative strategies required further questions in order to identify them clearly. Questions were rewritten in order to refine the diagnosis, and then retrialled. This process was iterated four times during the year.

By the end of the year, we had a diagnostic interview capable of revealing the knowledge and strategies that students use to solve linear equations. We also had data on the knowledge and strategies of a small number of students.

Findings

RESEARCH QUESTION 1: What knowledge and strategies do students use to solve equations?

Our initial classification of strategies for solving equations was based on the work of Kieran (1992), whose highly respected review of the learning and teaching of algebra described the strategies that students use. However, we added some further strategies that we had observed students using. We considered an *inverse operation* (1c) (see Figure 1) on a one-step equation to be different to Kieran's *working backwards* (3b) on multistep equations. It was difficult to distinguish the difference between Kieran's *known basic facts* (1a) and inverse operations (1c) as students often justified their answer by describing the inverse operation when, in fact, what they had done was use a known basic fact. One-step equations with larger numbers were therefore used to elicit the use of inverse operations.

We also found that the strategy of working backwards was not as homogeneous as had been assumed. Many students partly worked backwards and then used either *known facts* (3c) or *guess and check* (3d). When large or decimal numbers precluded the use of these strategies, they could no longer use working backwards. We also included the strategy of *using a diagram* (5). This resulted from our explorations of questions in context, where a number of students solved equations through direct use of diagrams.

Our final classification of strategies is listed in Figure 1. It should be noted that this is not intended to be hierarchical as we have insufficient evidence to make such a claim. In fact, 3c and 3d are clearly less sophisticated strategies than 3b, and at present we do not know the relative sophistication of 5.

FIGURE 1 **Classification of strategies for solving equations**

0.	Unable to answer question
1a.	Known basic facts
1b.	Counting techniques
1c.	Inverse operation
2.	Guess and check
3a.	Cover up
3b.	Working backwards
3c.	Working backwards then known facts
3d.	Working backwards then guess and check
4.	Formal operations/equation as object
5.	Use a diagram

(Based on Kieran, 1992; our amendments in italics)

RESEARCH QUESTION 2: What diagnostic questioning is appropriate for eliciting the knowledge and strategies used by students?

In a manner similar to that used in the numeracy development projects (Ministry of Education, 2003b), we found it useful to separate the questions into knowledge and strategy components. Students' uses of strategies were assessed by interview, as there was a need for supplementary questions. However, in the later iterations of the diagnostic tool, knowledge was assessed by a written test, guided by the research literature and our own experience. The focus of the strategy interview was on *how* students solved equations. In order to determine the most sophisticated strategies that students could use, we asked a series of increasingly complex questions. Our conjecture, that the full range of strategies could be elicited from students without using all the permutations of linear equations suggested by Herscovics and Linchevski (1994), appeared to be correct. Our easier equations were successfully solved by many students, and they used a broad range of strategies to solve them. In contrast, few students successfully solved our more difficult equations and only a small number of the more sophisticated strategies were employed. Initially we were not clear whether questions that were in context or purely abstract would elicit the most sophisticated strategies, and we used a mixture of the two. However, we found that many students performed significantly better on questions that were in context and, therefore, for the later versions of the interview, we produced parallel questions—in context and abstract.

Our knowledge questions focused on knowledge not routinely assessed in numeracy evaluations (Ministry of Education, 2003a). We assessed:

- using symbols and letters to represent an unknown
- manipulating symbols/unknowns (lack of closure)
- forming expressions with unknowns/symbols in them
- understanding of the equals sign
- operations on integers
- understanding of arithmetic structure
- understanding of inverse operations.

While it is debatable as to what constitutes knowledge and what constitutes strategic thinking, we considered the attributes listed above to be required knowledge for students to use some the previously listed strategies.

To summarise, we believe that the diagnostic interview we have developed is a useful tool for investigating the strategies used by students to solve equations.

Limitations

This project was always intended as the first stage in developing a framework for algebraic thinking. While we have successfully developed a diagnostic tool, the development of a framework still lies in the future. The diagnostic interview is of limited use unless we have knowledge of a progression of understanding. This can be obtained only by using the diagnostic tool with a cohort of students that is large enough for statistical analysis. The number of students we interviewed during this project was not large enough for such a purpose. However, it is planned to trial the diagnostic tool with a large number of students in 2007 and 2008.

A further limitation concerns the prerequisite knowledge for solving equations. The knowledge section of the diagnostic tool did not give us complete information about the knowledge required by students to use the strategies that we identified. This was because we did not attempt to investigate all prerequisite knowledge, as much of this is routinely assessed using numeracy development project diagnostic assessments. For example, knowledge of multiplication facts is required for using inverse operations. Also, whether the knowledge section of our diagnostic tool gave complete information on the additional knowledge required is not yet clear. Again, a much larger cohort of students needs to be investigated to give us data suitable for statistical analysis.

Building capability and capacity

Funding from the Teaching and Learning Research Initiative has enabled our research team to carry out collaborative research that extends the numeracy development projects into algebra. The whole team was able to engage with research literature and develop a diagnostic tool for early algebraic thinking. The research project has strengthened partnerships between the schools, the Dunedin College of Education (now the University of Otago's College of Education), and the franchise holder of Numberworks, Dunedin. It has also fostered greater research capabilities for all the participants.

We hope that this work will eventually lead to a framework for early algebraic thinking. It is anticipated that this will extend the Number Framework beyond the current upper level of advanced proportional thinking.

In 2007, two of the participating teachers from this project are involved in the Secondary Numeracy Project (see, for example, Hannah, Harvey, Higgins, Jackson, Maguire, Neill, Tagg, & Thomas (2006)). Another teacher will continue to lead numeracy within her school, though the school is not officially participating in the numeracy projects. The lead researcher has presented the findings from the project at the National Numeracy Conference in Auckland (Linsell, 2007) and we will be involving many more teachers in the next phase of the project. This plan has strong support from leaders of the Secondary Numeracy Project. Three of the teachers are keen to be involved in the next phase of the project and are determined to use the diagnostic tool we have developed.

There has been long-standing interest among local teachers about the difficulties experienced by students in learning algebra, and four more schools have already expressed interest in being involved in the next phase of the research. The merger between the University of Otago and Dunedin College of Education has created enhanced conditions for research and it is anticipated that involvement in this project will lead to further developments of research in mathematics education in Otago.

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Lead author and co-researchers

Chris Linsell is a senior lecturer at the University of Otago's College of Education, where he is subject leader in mathematics. He previously taught mathematics and physics at the high school level and has also worked as a researcher in biological psychiatry. His research interests are in the field of mathematics education, and he has recently completed his doctorate in algebraic thinking. His current research, planned for 2007 and 2008, is an extension of the project described here, in which he hopes to establish a framework for algebraic thinking in the domain of solving equations.

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