Appendix A: Anne Blundell, Auckland Girls' Grammar School

Proof-o-phobe to proof-o-phile

Proof was a simple and obvious topic choice when asked to focus on a weakness in my personal mathematics understanding. This weakness had undoubtedly transferred to my own students through my teaching. Proofs were not taught during my schooling. They were shown and discussed briefly, usually to the more able students. A repetition of this situation was apparent in my own mathematics classroom. My students were not being exposed to proofs, due largely to the combined reasons of proofs not generally being assessed, and the lack of confidence of their teacher.

During university education I struggled with proofs and came to see the art of proof as "memorisation without insight or understanding". Even when patiently explained, I could not create a comprehensive explanation of the manipulation or insights that led to replicating the proofs I encountered.

It was with trepidation then, that I set out to correct a personal weakness this year, with the aim to increase my own confidence and at the very least, expose the art of proof to my own students. Having not witnessed explicit pedagogy around exploring or developing proofs for student learning, I looked forward to the researchers and teachers helping to shed light on the ability to generalise and specialise at the same time, as well as provide sound pedagogies for scaffolding students' ability to "do" proofs.

My learning process

The process initially began with background reading "around" the subject. This was quite different from "doing" proofs. The researchers provided several easy-to-read relevant chapters from different levels of the curriculum. Extensive older mathematics texts at Auckland Girls' Grammar School also provided a treasure trove of elegant proofs. A trip to England and the array of mathematics books for nonmathematics specialists there also proved helpful. Ian Stewart writes in a simple, straightforward manner about the art of proof, and its crucial underpinning of mathematics. Stewart's book, *A Letter to a Young Mathematician*, took proof out of a mathematical context using words and letters (Ship-to-Dock Proof). This became the first proof I attempted, at least three months into the project! However, I could not make the vital connection necessary to solve the proof. I was concerned by this old familiar obstacle. I discussed my findings with the researcher who provided the idea of using manipulatives to get a better "feel" for the problem. This form of pedagogy was in line with my belief in constructivist pedagogies for students and re-ignited my love of manipulatives in the classroom, but on a topic with which I had not previously thought to use them.

Mathematics colleagues in my department were also inducted into helping me progress my understanding of proof. The pure mathematicians proved an invaluable resource and patiently explained many proofs where the middle (key) part was particularly "muddled" for me. Reasoning for steps in a proof had often totally escaped me. A step missed in a proof frequently proves crucial to understanding. The central steps often make connections across branches of mathematics and this aspect of my thinking is still a "work in progress".

Websites using applets also allow the user to obtain a better "feeling" for the generalisation that needs to take place were also shared as part of the research project. The applets proved illuminating and made me think about special cases, counterexamples and algebraic generalisations when constructing a proof.

The research project gave impetus to focus on several aspects of my learning that invariably "cross-pollinated" with my mathematics teaching. These included actively searching out proofs to investigate or to "do", thinking about underlying mathematical proofs and their history that were central to the topic being studied in class. Also included were different approaches to proofs; for example, geometric versus algebraic proof, increased dialogue regarding the development of proofs. Pedagogies for better learning and encouraging student exploration of proof, rather than simply being a passive observer, were also used.

At the beginning of the year I had informed my classes that I had chosen an area of personal weakness to focus on as part of a research project. As the year progressed I found I began to believe in the advice I gave my students "that the ability to do mathematics is 99 percent confidence"! When asked by a student in July, I produced my first statistical proof in front of students, in class. This was a major achievement for me, as in the past I would have taken it away and asked a mathematician.

Learning as a group

The teachers and researchers involved not only provided ideas, advice and guidance. They also provided motivation and inspiration. At group meetings I did feel that I was the only one who had chosen a weakness in mathematics, whereas others had chosen mainly topics they were interested in. I could see the benefits of this in their progress and understanding about how mathematics had changed their fields of interest; for example, logarithms and differential equations used in medicine. I felt this difference in topic choice acutely at the NZAMT presentation, beginning my presentation, admitting to peers that I felt I had failed my own students by not exposing them to proof, only in the most vague sense. However, feedback from the presentation made me even more aware that teachers are learners too. This desire we have to understand and improve learning needs to be modelled to our students in the classroom.

Impact on students

I found when teaching that I would not settle for anything less than a full explanation, this being the fundamental reasoning behind proof! This led to more cross-pollination between the professional developments that I undertook this year. For example, in ICT training on Excel I developed a self-marking worksheet for the generalisation of the quadratic formula. This came about from students' desire and requirement to understand quadratic patterning and how to find the coefficient "*a*" by division by 2 of the second difference. The spreadsheet was included in Moodle, the school intranet system. Other ICT resources focusing on proofs were added for the students to access.

The largest impact on students was their teacher setting aside time for class discussion of the grey boxes that contained the majority of proofs in textbooks, and not avoiding them.

Two of the reasons girls in particular do not continue to higher levels of mathematics learning is seen as the abstract and humanly isolated nature of higher level mathematics, proofs in particular. This could be alleviated in my opinion with progressive exposure to algebraic proofs prior to tertiary study.

Lessons were started with proofs obtained from the book *Cooperative Learning*. Class sets of proofs were cut up and laminated so that students could work in pairs. The students would have to discuss the steps involved in a proof. They would then write up the proof, annotating each step with the algebraic manipulation and reasoning beside the line of proof. This also had the effect of increasing and reinforcing students' algebraic skills. The focus also led to wider vocabulary being used in the classroom, such as logic, reasoning, generalisation and counter-examples.

Reflection on the project

This is my seventh year of teaching and has been another interesting year of learning and teaching. The presentation at the biannual NZAMT conference was perhaps the highlight. Work was presented from each group's research, to our peers from around the country. There was a great deal of interest in extending mathematical communities beyond the school environment, as mathematics teachers require mathematical spurs and inspiration to keep their mathematical fires burning. The project has kept the focus of proof at the back of my mind for the entire length of the project and probably beyond that! When opportunities have arisen it has been brought to the front of my thinking to better enhance mathematical learning for my students. The Mathematics Department at Auckland Girls' Grammar School has had a year of informal discussions on the merit, development, history and relevance of proofs to today's students. My students have been exposed to proofs, they had to think about how proofs work and the important role they play in underpinning mathematical knowledge. A further highlight was the inspiration received from the other members of the group to further progress learning and understanding and to ways of making mathematics we teach more relevant to students.

This project has highlighted again the importance of teachers being researchers in their own classrooms, as an effective means of modelling self-learning to students as well as increasing their knowledge about what works in education. This is the fourth research project I have been involved in. Projects such as this are the reason I will finish my Master's in Mathematics Education in 2008 with the aid of a PPTA study award and also a reason I will continue to teach. I look forward to being involved in many more.

Appendix B: Linda Crisford, Westlake Boys' High School

Learning more about trigonometry

At the beginning of the year I was asked to join a group of six other mathematics teachers looking at how we continue to learn ourselves and how our learning improves students' understanding and results. We had to choose one area to focus on. I found this difficult as there was a lot of choice and a lot of areas I was interested in. took up a suggestion from Judy of trigonometry as this was one of my weaker areas.

I began by first asking teachers in my department what they felt about mathematics. Their responses were most illuminating. They felt quite strongly that there was a loss of mathematical experience and in-depth knowledge of specific concepts; for example, trigonometry and aspects of calculus. The older, more experienced staff felt they were too tired from teaching and the extra demands a school places on them (for example, sport, meetings, pastoral duties) to be able to work with others in the department.

I have always felt that to continue to be enthusiastic and gain more meaning and understanding and to develop as a better teacher teachers need to keep learning themselves. But when I asked other teachers about this, they said that "mathematical stimulus in schools for teachers is lacking". Where has the passion gone?"

I then asked how they experienced mathematics when they were younger. Here are their responses:

- "Maths gives you kicks from something that is unreal."
- "Pure mathematics is awesome as it provides the rules that end up being the tools that are applied to real-life situations."
- "I got a kick out of living in a completely abstract cloud nine world."
- "Mathematics nurtures your soul—it's like music."
- "Mathematicians are artists."
- "I loved mathematics because I was good at it."
- "I found mathematics questions challenging and I wanted to find out the answer."
- "Mathematics has clear defined routines and settled me as a person. It helped me develop self-discipline."

And the response I personally liked best was "Mathematics was a turn-on."

This led to a discussion about how we could still feel these positive exciting feelings as we grow older and how as mathematics teachers we could continue to upskill and further our own education. We discussed the importance of this with respect to the large numbers of mathematics teachers leaving the teaching profession. Because of these discussions I ended up with an additional focus on the special nature of learning mathematics as teachers and as students, although I was supposed to be working on developing my skills in trigonometry.

When trying to get help from other teachers in my department for my learning, I found that they were happy to assist me so long as I was fast (because they are so busy). I would say, "Can you just show me how to do this in four minutes. If I don't get it in four, forget it, I'll come back another time." Of course this meant I usually got seven to 10 minutes of one-to-one explanation. I also asked if I could be a silent part of their class when they were teaching. This was marvellous for me because I was safe from embarrassment: if I didn't understand everything it didn't matter because no one was going to ask me to answer a question. Lights continued to come on for me as I sat in on Scholarship classes with two colleagues. I wished I had more time for this, because being a student yourself is important; remembering what it is like to be on the other side.

I spent time with a colleague while he explained personal ways of teaching trigonometry. These sessions I recorded and used in my teaching practice. They enhanced my personal learning and enabled me to teach more effectively and confidently.

I also decided I would research trigonometry myself to see if I could educate myself. I used books and the Internet, and worked hard. However, when I could not understand a concept or problem and could not move forwards, the importance of teachers to unpack the learning was brought home to me again. So I believe, from experience, that specialised teaching skills with teachers who are highly academic are needed to promote excellence in mathematics learning. I believe strongly that mathematics is learnt through communication and discussion with other people. It doesn't matter how much you read, or try to do the exercises on your own, you need a "teacher" to explain. It doesn't matter whether the networking is between child/child, student/teacher, teacher/teacher, parent/child or friend/friend. This "teaching" is a key to developing confidence through shared experiences. This has been a notable step in my learning as I realise that teacher input is crucial to further understanding and the development of more in-depth concepts.

For teachers in the New Zealand school system, colleagues can help to provide connections to what you already know and new skills that need to be acquired. My colleagues proved invaluable. They helped me take more frequent academic risks and develop greater self-confidence, which led to an upward spiral of not only my learning but that of the students I teach as well.

It takes a lot of confidence to ask your colleagues for help and expose your weaknesses to them, so a supportive community is very important. We are lucky at my school to have some brilliant mathematicians who have helped me in all areas of mathematics. The fear of the unknown and some problem "looking hard" can deter you from asking for help. You do not want to look an idiot, especially when you are experienced and are supposed to know it all.

But time is so against teachers as our commitments are enormous and finding time to extend your own learning can be difficult. Nevertheless, just a few minutes here and there can make an enormous difference.

Another teacher in our department was having difficulty teaching basic trigonometric functions. She was stressed, and said that there was no time to collaborate with her peers to go over teaching strategies and knowledge. I said, “We’ve got five minutes, let’s go.” In that time I explained a technique for drawing and understanding trigonometric graphs. Tracey reported back that this worked very well. We subsequently decided together to further discuss the problems of teaching trigonometry, and to ask another colleague to help us.

The importance of sharing learning was brought out again for me recently. I had a discussion about learning mathematics with the 16-year-old daughter of a friend. She explained that after two good teachers her present one tells her class to read notes for themselves but does not explain them. Furthermore, students are not permitted to talk to each other in class. She then said that she got around this by working with her mother on the problems.

The result of my year has been the realisation that it is important to build a mathematical community whether it is with our friends, colleagues, between school communities or between primary, secondary and university teachers. Professional development needs to be fun and regular and about ourselves to keep ourselves learning.

Next year I hope to continue working with the university team, particularly on networking between schools and linking learning through primary, intermediate and secondary schools. I hope to be visiting different schools to gain a deeper perspective on how mathematics is taught elsewhere. I would like to help ensure we can keep mathematics teachers in education for future generations.

Appendix C: Margaret de Boer, Tamaki College

Learning more about logarithms

How I chose my topic

I found choosing a topic very easy. I was teaching a Year 12 class at Tamaki College. I had taught this level for several years and had implemented Level 2 NCEA mathematics at the school. When teaching logarithms I always felt I wasn't giving the students enough depth for the subject.

The textbooks seemed to focus on a set of rules for logarithms, which the students learnt and then practised. Also, Unit Standard 5246, taught as a backup for the Algebra Achievement Standard, made me want to teach logarithms as a key skill. The students learnt by rote and practised until they got it right. Assessment was straightforward: one question for expanding, first for indices, then for logarithms, and so on. But the students hardly ever passed this assessment. I put it down to the students not doing their share of the work, and believed that they would have success if they tried harder. Over time, however, I could see the students sometimes having success grasping quite difficult concepts, and yet they still could not learn these three or four little rules. I was considering spending a bit more time teaching the topic, and less time giving the students practice to see whether that would help.

Other reasons for wanting to research logarithms were that I felt there are probably many applications, and that I didn't understand logarithms well enough myself. The students come across them in statistics and modelling, as well as calculus, so logarithms are important, and here I was just getting them to learn rules.

My learning process

From the first two group meetings, I got a lot of good ideas and felt inspired listening to the areas that other people in the group were covering. I was especially attracted to the topics of modelling, and history of mathematics. I remembered how much I had enjoyed doing the history of mathematics while getting my degree, and reflected that now I was always too busy to follow up on anything. I wanted to incorporate these into my research. As it turned out, I spent a lot of time on the history of logarithms, and less time on modelling with them.

Straight away I got out my old history of mathematics book, which I had fondly carried with me for the last 10 years, but had rarely opened. There were always too many things that were more urgent and I couldn't afford the time.

After reading about the history and origins of our present day number system, I tried to show some of my students that concepts that we took as given were actually a man-made convention. I incorporated the Babylonian place value system into a lesson. It was my intention also to convey to students how much time had gone into developing concepts that we covered in class in half an hour.

I also explained to students how, although the wider applications of mathematical ideas grow as time goes on, initially the discovery of new concepts was driven by financial gain and attempts to make human endeavour more efficient. For example, geometry developing through surveying and hence being described as “the gift of the Nile”.

This was not a major part of my teaching: I just included my thoughts for a short amount of time for some lessons. The students often focus on examinations, and if they are asked to learn anything that is not directly examinable, they may offer complaints or become dissatisfied. However, when Judy came to visit, she reminded me that interest is contagious, and I thought how some of my enthusiasm might have a positive effect on the students. Maybe my disappointment with many of the students not passing the topic had also been communicated to them without my realising it.

I spent a lot more time thinking about the way I taught than I did changing the lessons.

Learning as a group

One of the main changes that this project has brought to my teaching is that it has allowed me to have subject-based discussions with my colleagues without feeling guilty.

It was really nice to have someone interested in what I was doing in the classroom, but more than that, the feedback was essential to find out if what I was doing was sensible. I make a real effort to stop what I am doing and listen, discuss and exchange viewpoints when someone in the department takes the time to talk. And then I am able to bounce ideas off them as well.

Impact on students

My aim when starting logarithms with the students was to get them to connect with what drove the originator to discover them. Hopefully that would get them to appreciate how important they were and they would see a use for them.

The students had quite low algebra skills and I used a revision of the indices laws to start the lesson. This also gave the something familiar to connect with. We had developed $a^n \times a^m = a^{n+m}$ earlier on, but I had reservations as to whether they remembered them. We first generated 9, 27, 81 as powers of 3, and then tried to generate 10 as a power of 3. Using trial and error we took seven attempts to get an index value that gave us a good approximation. Not all students were completely sure about the steps we followed, but they did accept what we did as valid process.

After a discussion about generating any number as a power of 3, the students chose to try to generate 1.4 as a power of 3, and they went through the same process. Some were successful in finding the power.

Next I spoke to the students about John Napier and how it was the practical need for a more time-efficient method for dealing with multiplying large numbers that drove him to invent logarithms.

Once again I tried to get the students to connect with what was familiar, this time using "10" as the base. We looked at 5 x 3, and students checked the powers of 10 I gave them for these numbers, and then formed the product. I felt that the students were not really following the process in the same way I was, but they did get something from it. I don't believe they were thinking about the last equation in the same way I was.

The form of the logarithm rules at least must have seemed familiar after this exercise, because they were very good at practising them after that. They even did some for homework. I wish I had found a better way for teaching them the laws of indices, as that seemed to hold them back still, but they got straight into practising the questions, much more than they usually would. Previously, they would just do the bare minimum of work required, but this time they worked through whole exercises. With the reduced level of algebra skills in the class, this practice strengthened their algebra skills in other areas as well.

Reflection on the project

The main difference that the project has made to my teaching is that I believe there is a way to help the students to succeed in mathematics. I used to spend a lot of energy trying to motivate the students to take responsibility for their learning and do more exercises, with not that much success. I believed the students at the school would gain better academic results in the long term, once a kind of threshold of student engagement was reached, when it became cool to be successful in class.

I still believe that will happen, but I also have more faith in raising student achievement in the short term. I am more prepared to go that extra mile to find a way that works for the students to understand the topic. I see taking the time to read around a topic, and engage in more professional discussion, as a way of doing that.

By starting with something familiar, and working towards something new in steps, all the students will be already engaged for at least some of the time. And then when they are asked to participate themselves, will be more prepared to do so, even if they see the work in a different way from the teacher.

Thinking about student learning in this way allows me to be more comfortable with where the students are at, and less anxious about what they don't know and aren't familiar with. Hopefully this will empower me to let students grasp the big ideas without sweating over the little details.

Appendix D: Anna Dumnov, Senior College of New Zealand

Using history for teaching and learning mathematics

Introduction

One of the major challenges every mathematics teacher has is how to make the students enjoy mathematics and how to demonstrate its relevance to the everyday life. I have always been aware that using history of mathematics as a pedagogical technique can help to interest and involve students in their learning. According to Ian Stewart it is astonishing how many people believe that "mathematics is limited to what they were taught at school", that "the answers are all at the back of the book" and that "there is no scope for creativity and no questions remain unanswered" (Stewart, 2006, p. 34). I strongly believe that it is important to tell the students that there is mathematics outside the textbook.

The history of mathematics is full of examples. History shows how mathematics grows and develops and indicates "that the mathematics of a few years from now will no doubt be different from ours while still including today's ideas" (**NCTM,** 1969, p. 4). I believe that "putting mathematics into a cultural context, explaining what it has done for humanity, telling the story of its historical development, or pointing out the wealth of unsolved problems" (Stewart, 2006, p. 37) will trigger deeper learning of mathematics, which will, in its turn, provide more opportunities for asking new questions.

Being aware of this, I often refer to historical facts that I know and make connections with the developments of past when teaching. However, I find my own knowledge quite limited and I always wished to learn more myself. Thus when offered an opportunity to be involved in this programme, where the focus was to be on a personal area of interest, I had no doubt that I would like to focus my own learning on the history of mathematics.

Introducing the history of algebra to Years 11 and 13 classes

While preparing to teach linear algebra in Year 11 classes and evaluating the previous experiences with this group, I decided to change the way I taught this topic and to introduce it by referring to its historic context. Some of the possible questions to be discussed were where the word "algebra" came from, when did this happen and how did algebraic notation develop?

I followed this plan and spent time talking about the Babylonians and how old algebra actually is. I also told them that "algebra" could be translated as "science of equations". It was a start. Although students did not seem very excited, I believe that it was a better start for them than the traditional textbook approach. My intention was to continue with this approach with the Year 11 group into the teaching of quadratic equations later in the year.

At approximately the same time, I was teaching the factor and remainder theorems in the Year 13 class. I talked about the "science of equations", about the history of solving quadratics and cubics from Al Khowarismi to Cardano (Nahin, 1998). As one of the students noticed, we did all the mathematics from 800AD to the 1500s in one hour!

The Year 13 students were much more interested than the Year 11 students, maybe because I had taught most of this group the previous year and they are used to me going on small excursions into history from time to time.

Teaching logarithms in the Year 13 class

Several weeks later I started teaching logarithms in my Year 13 class. I had no intention of incorporating history into my teaching and I was teaching it in the usual academic approach starting from the formal definition, then introducing the logarithm to base 10 and then the logarithm to base *e*.

As students are used to working with the decimal numeration system, the decimal logarithm did not trigger much special interest. However, this was the first encounter with the number *e* for them, and at this stage they knew nothing about it. As this group is used to my incorporating historic developments into the lessons, it was only natural that they expected me to explain in detail why we used this strange number and the questions "What kind of number is this?", "What is special about it?", as well as "Who was the first to think about it?" were immediately asked.

This situation caused several points of difficulty for me. It was very difficult to explain at this stage the significance of *e*, as I believed that it is very closely related to calculus with which the students were not familiar. Furthermore, I was not familiar with the history of *e* myself. I knew only that it was Euler who formalised it in the first half of the 18th century.

I had to tell the students that *e* is an irrational number, widely used in calculus because of one very specific property, namely the very special behaviour of its derivative. I had to admit that I knew very little about its history, and suggested that we investigate further together.

At first my personal investigation took me to several textbooks which, although introducing the logarithms of base *e* at the early stages, all suggest waiting until the later chapters on calculus in order to find out about this strange number. Later in the chapters about calculus, *e* is explained as the base of the power function for which the value of the gradient function is the same as the value of the function at every point.

However, further investigation showed that this number was known to mathematicians at least half a century before the invention of the calculus (Maor, 1998). The property of *e* being the limit of the compound interest formula was known as an observation before the concept of limit had actually been developed by mathematicians. Similarly the connection between this number and the area under the basic rectangular hyperbola was also noticed before the formal invention of calculus. Thus my readings took me to pre-calculus history and my personal learning expanded to include what follows.

My personal learning

Napier's logarithms

My personal learning involved understanding *e* from three different points of view. The first was the development of logarithms by Napier, who, with an interest in efficient computation, used existing knowledge of exponents to devise tables that would help with large computations. The tables subsequently were developed using *e* as a more efficient base, although Napier did not do this himself.

The second point of view was that of the mathematics of finance, in particular the compound interest formula. The number *e* emerges from this formula where the number of times an amount is compounded during a year tends to infinity.

The third area of learning was that of quadrature—the process known since Archimedes times and developed by mathematicians such as Kepler, Fermat and Descartes—finding areas under curves using successive approximations of polygons of known areas. The area under a hyperbola was resistant to this method until Saint-Vincent and his student Alfonso Anton de Sarasa realised that it could be done using the logarithms of distances on the *x*-axis. This led to the idea of the exponential function as the inverse of the logarithmic function.

As I was reading and learning, I shared my new knowledge with class, stressing at all times that I was in the process of learning myself and asking students to participate and contribute to this learning. I wanted students to see that not only was I learning new historical information, but that this information led me into a new mathematical context. It was an opportunity to show the students the mathematics outside the textbook, and that the more mathematics is learnt, the more opportunities one will find for asking new questions (Stewart, 2006).

One student's journey

There is a tradition in our school that our academically top students make a 30-minute presentation to their peers on a topic of their choice. It was very pleasing for me when, following our discussion in class, one of those students approached me about doing her investigation into the nature of the number *e*.

The student knew nothing about *e* at this stage, apart from differentiating and integrating the logarithmic and exponential functions. She sought to develop historical issues (the long history of *e*; contributions by different mathematicians; the logarithmic slide rule), applications (logarithmic spiral and its appearance in nature; architecture of the catenary; the link with the normal distribution) and new ideas (transcendental numbers, series expansion of *e*; area under the hyperbola; Euler's formula).

It was a very interesting experience for both of us. We met several times and discussed the ideas, the readings and the structure of the presentation. It was very interesting for me that her readings led her to a different approach to the number *e*. Eventually the student decided that she wanted to organise her presentation as a "mystery adventure". She was very concerned about being interesting, accessible and inclusive for all her future listeners. The final talk was a success, and a suitable culmination for my own investigation.

Evaluation

Ian Stewart points to the existence of the old conflict of learning "for exams" and true learning. "Putting mathematics into cultural context, explaining what it has done for humanity, telling the story of its historical development, or pointing out the wealth of unsolved problems … leaves less time to prepare for exams …" (Stewart, 2006, p. 37). This conflict cannot be resolved externally. Each teacher makes the decision for himself. It is hard to overestimate the value added to my teaching from the experience described above. Not only have I learnt new historical facts and new mathematical context, but I was able to share it with my students and colleagues within my school, as well as with a wider group of teachers. I believe that I established a special relationship with my students and by my own example have shown them that learning is never really complete and is definitely not finished when one leaves the classroom. For the student who undertook that journey into the history of *e* there were also several very valuable outcomes. She obtained: new mathematical knowledge; new experience orally presenting material to her peers; ideas about connections within her mathematics course; and personal experience of "… the more mathematics you learn, the more opportunities you will find for asking new questions …" (Stewart, 2006, p. 37).

There is no doubt that the history of mathematics is a powerful pedagogical tool. I would love to do more in-depth theoretical research into the pedagogical tools and values of this approach in the future, as I will continue to incorporate this approach into my teaching. It can help students to see that mathematics is not static but an evolving body of knowledge. It shows the relationships between the different parts of mathematics and helps to develop an understanding of what modern mathematics should really be. History shows that contemporary mathematics is a mixture of much that is very old with newer concepts, such as sets (NCTM, 1969, p. 17). Insight into historical developments can inspire students, stimulate their interest in the subject, stimulate in-depth research and help their engagement in further learning.

I feel that I greatly benefited from participating in the project and I would love to be able to participate in similar projects in the future.

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Appendix E: Jason Florence, Otahuhu College

University algebra to probability

A diary

Calculus Day Nov. 2006: At the project meeting everyone chose what they would like to look at. I was interested in looking at university-level algebra and the role it plays in the real world. I got some positive feedback from the group and was pleased with the decision I had made. I chose to investigate this as I felt it could have a positive impact on my teaching. I saw it as a gateway between school and university.

1 Feb. 2007: During this meeting, we consolidated what we had decided. It was an enjoyable experience although it did seem a little strange being video-recorded. During this meeting it was confirmed that I would attempt to attend the MATH 328 lectures on algebra and applications by Arkadii Slinko.

2 Feb. 2007: After reviewing my school timetable I decided it would be too difficult to attend lectures in person. This was a big disappointment as I believe that by attending the lectures I could have got a lot from the course.

7 Feb. 2007: Arkadii emailed me the course notes from 2006. There were to be no significant changes to the course. The course consists of: Integers; Cryptology; Groups; Fields; Polynomials; and Codes. I have touched upon all these topics at one time or another although several only at a very low cursory level.

I decide on the following timeline:

Term 1. Readings and trialling at least two lessons based on Chapter 1 and Chapter 2.

Term 2. Reading and trialling at least two lessons based on Chapter 3 and Chapter 4.

Term 3. Reading Chapter 5 and trialling at least two lessons.

Term 3 (holidays). Presenting my current progress at the NZAMT conference.

Term 4. Reading Chapter 6 and trialling at least two lessons with my junior classes.

December to January. Write report on what I learnt, what I observed and what I conclude.

18 Feb. 2007: Chapters.1 and2, Integers and Cryptology. There are 24 sections so I aim to cover three per week.

1. Natural numbers. This chapter had an elegant induction proof of natural numbers.

2. Divisibility and primes. This took a bit longer to get through, as I had to get used to reading mathematically again. I had not done this for quite some time and it was a challenge to get back into it. This chapter had a very nice proof of the fundamental theorem of arithmetic. I also appreciated the prime number theorem, which can be used to determine the number of divisions required. This formula gives an asymptotic approximation due to the irregular occurrence of primes.

3. The Euclidean algorithm. I didn't understand this first time around at university. I recall having one of those very theoretical teachers at the time but after reading through the notes I got it almost straight away. I think this algorithm could be something I could try with students.

25 Feb. 2007:

4. Euler's ø-function. I did not recall this function from my university days. It is relatively straightforward.

5. Congruences. Euler's Theorem. Looked at some lemma's and then a proof of Fermat's little theorem (luckily it wasn't his last theorem), and Euler's theorem. I can use this theorem but I still have not convinced myself I understand the proof fully.

6. Intergers modulo n. This moved into more abstract symbolism. The section then goes into modular multiplication, which I feel a have a good understanding of as well. I then read about the multiplicative inverse of an element. While I could introduce modulo arithmetic to pupils, I think it would be inappropriate and would cause more troubles than enlighten.

3 Mar. 2007:

7. Representation of numbers. Classical decimal positional system. A quick look at our current decimal system. A historical look at other systems, most notably, base 60. Now, base 2 is even more important than base 10, due to the emergence of computers. Representations for real numbers, not a lot here really, except that the negative powers of 10 are used to express those real numbers which are not integers. (I never considered it that way.)

4 Mar. 2007: Assignment 1. Attempted Questions 1 to 6. Question 7 required a computer application I do not know how to use. Question 8 seemed to be in another language, perhaps highlighted in the course. Not all correct.

8 Mar. 2007: Had an awesome lecture on probability in MATH 708. I'm going to try the activity with my Year 13 statistics class.

10 Mar. 2007: Chapter.2, Cryptology:

1. Classic secret-key cryptology. A quick introductory section on why cryptology developed. Sadly, very boring, which is a shame as this should be an interesting and informative component of the course. Secret-key cryptosystems. Pretty standard stuff. The one-time pad. An old system that was invented in 1917. An affine cryptosystem. This system is much like the one-time pad system and once again is based on modular arithmetic (base 26).

15 Mar. 2007: Lesson 1, using knowledge from MATH 708 with my Year 13 statistics class. There is a move in probabilistic teaching that students should link the theoretical with the empirical. Students start with an intuitive feeling for the situation, and then they should move to an empirical understanding (such as running an experiment or simulation), and from these results deduce a theoretical model.

I presented the class with a Hospitals and Babies scenario. They began with an intuitive guess, and then we ran a simulation. This was a hit with the students.

18 Mar. 2007: Hill's Cryptosystem. This system uses matrices. I thought that this system was quite elegant and I created my own code and tried it out and it worked. It would be something to consider showing more advanced students but I do not have a class that could understand this method. Modern public-key cryptology. One-way functions and trapdoors.

3 Apr. 2007: I was working with my Year 12 class and we came across the quadratic function. Years ago I had attempted to prove this to a Year 12 class at a different school but skimmed over the proof. Normally I would stay away from this proof as I was a little shaky on the whole completing the squares idea. I decided to review this concept and once again attempt to show it to my class. I went back to my first-year text. It all looked so easy (compared to the Level 3 stuff I had been reading). I worked through the parts I had difficulty with, and then prepared and presented a proof to my class. While I was well pleased how this went I think it all went over my students' heads. I still felt it was worthwhile.

10 Apr. 2007: I'm beginning to have serious doubts about this paper, and will need to think of another way to connect higher learning with the classroom and my class. A possibility could be to look at where maths occurs but it seems a bit light.

Term break—not much done here.

11 May 2007: Two weeks back at work and I have floundered a bit with the algebra. It has been very difficult to continue with the learning without regular support. I have been unable to attend the lectures, and while the topic is quite interesting I cannot see how it is impacting on my

teaching significantly. I can start to understand how students can get frustrated by lack of structured learning. Despite this setback, I am still quite committed to this project, despite having a month's break.

16 May 2007: Had a good meeting with Judy and highlighted some concerns and we discussed some solutions to get the learning back on track. We look at taking one small aspect of what I had studied (modulo operations) and perhaps develop some interesting activities with a top Year 9 class. Judy also gave me some suggested readings I will try to follow up.

18 May 2007: I have been writing a paper about stats education. I think at the moment, statistical education is playing a significant role in my professional development at the moment and it seems silly to ignore the good learning I have done on this paper. I have done some really good stuff in this course and I believe that this is my best avenue given the time and resources available to me. I hope to set a plan during the week.

21 May 2007: After reading through my notes I have decided to go down the probability track. I have already done two readings for the course and have access to other readings. This paper looked at the underlying reasons why people make certain decisions and factors that can affect your decision-making process. From the other reading what I gather is that people make bad calls due to a lack of understanding about probability.

27 May 2007: Have done another reading that highlights the need to educate students with respect to misconceptions and error encoding that they bring to the class. All these readings are really good and interesting. But I'm faced with the problem of how to bring this to the classroom and at what level.

2 June 2007: A reading really highlighted what is called heuristics and how they affect our decision making. Essentially what has been shown is that people have preconceived notions of probability and these notions create personal biases or fallacies that act as a trigger to decision making. The three main heuristical concepts are representativeness, availability and anchoring and all these will lead to incorrect decision making.

10 June 2007: It's really clear that I cannot get these into the classroom in an effective manner unless I set aside a lesson for an idea. I tried Monty's dilemma with a Year 10 class and it fell kind of flat. I discovered that I could not easily break down the students' adjustment heuristic. It was really funny and the class and I had a great debate and disagreement. I know a couple of kids came around to understand what the problem was. I'm still not too sure who I want to bring this to. With the Year 10 class, I tried to investigate the heuristic in the last 10 minutes of the lesson. It really needed a whole lesson and to be better prepared than I did.

15 June 2007: I have been attending a workshop looking at statistics. These run on a Thursday from 4.30 pm to 6.30 pm. They are not really that useful for this course but it's interesting to see what is happening and it's giving me ideas about new approaches to teaching statistics. I wonder if there are math teachers groups that meet weekly or monthly and discuss maths issues.

17 June 2007: Another reading done. This highlighted some four issues that children bring to the classroom, loosely called Fairness, Luck, Equiprobability and Randomness. From the readings I have done I think there is enough to develop some kind of investigation involving probability and student (and adult) misconceptions and misunderstandings.

24 June 2007: Another reading, this time about uncertainty. There was some good stuff in this reading about the error coded concept of percentage. The research also showed that a lot of pupils believed that 50 percent meant that "you cannot tell", "it doesn't matter" and regarded a 50 percent chance of an event occurring as "you just don't know". I think I now have a good plan of attack: set up a 1–3-week probability module; run some type of an investigation(s) with the class; record what happens in each lesson; critically reflect.

As I am Year 9 co-ordinator, I will set this up for the Year 9 course in Term 3. I am confident I have got something really good here. It's going to be a big change from what I started looking at but it has already had a huge impact on my way of thinking about how to teach probability and it will have a massive impact on what I do in the classroom. I wonder if I should do this just with my class or for the whole department.

26 June 2007: I really need to get the planning of each lesson done during the holidays. I must plan out this module carefully. Should I start with heuristics or fallacies and misconceptions? I don't think a module has ever been taught like the one I'm planning in a New Zealand school before. I wonder if I can videotape any of the lessons. I think they will need to be fairly scaffolded though (remember the Year 10 attempt). It will need good visual impact so some PowerPoint lessons would be good and some experimental equipment (maybe dice and spinners). Perhaps even introduce using the random number generator when we look at randomness.

27 June 2007: Meeting with group. How are we going to present at NZAMT?

2 July 2007: Well, the meeting could have gone better from my perspective. I received a rather cool and frosty reception for my turn of events. It looks like I pretty much blacklisted myself from the workshop presentations, which is a bit of a shame as I believe that I would be one of the best presenters. I am learning about stuff I never knew existed and in that regard I feel my contribution would have been ideal! As I've said before, I've done the theoretical side of this research and now I need to do the hard work and create a module based on what I have learnt.

22 July 2007: Have made good developments with my probability project. I am pleased with the overview of an eight-lesson module and I should have something cool to hand out at NZAMT. I have adapted and copied Jane Watson's Literacy Statistics survey, and given it a probability slant.

2 Aug. 2007: Group meeting, preparing for NZAMT. My contribution will be my story. I started working as an individual, using notes from university. This failed, so I went back to a group dynamic. Mathematics is not done individually in the real world.

12 Aug. 2007: The meeting went well and I'm back in the presentation so I'm well pleased with that. The probability module is going well and I have developed some quite cool lessons. I'm also putting together a big booklet (over 50 pages that contain my research and lessons). I got some neat video footage from Yoko about the Monty's dilemma problem which is awesome. It was decided that we are not going to give out stuff at NZAMT, but rather focus on what we have been doing and our journeys. I really think this is a good idea (mainly because I am still working on the probability module). My biggest challenge now is putting together what I have been up to this year and encapsulating into 10 minutes with a focus on working in a group.

7 Sept. 2007: Practice presentation to Mathematics Department at Auckland University. It is one thing talking in front of your class or your peers, but it is something quite difference trying to present in front of experts in the field. I was just hoping I didn't come across as speaking a load of nonsense. I think the presentation ran together quite well and I was pleased how my section went.

14 Sept. 2007: Got some feedback from Judy. The department was generally pleased with our presentation. Judy highlighted that she was hoping I talked more about our group than the other group I worked with. Each group influenced me in different ways. My earlier group where I learnt about heuristics helped me develop and learn about this branch of probability. The second group I worked with, this group, really made me pursue the idea of developing what I had learnt and make it into something useful for teaching. It both enhanced my understanding of heuristics and made me work to developing something of a high quality that teachers can use.

27 Sept. 2007: Conference day. Our presentation went really well. We had a full house and everyone did an awesome job. I modified my part on recommendations from Judy and I think this helped my section a bit.

Summary: I have to say this has been an amazing journey for me. It started brightly and then took a dive. It then did an about face in a totally new direction and reached further than I thought it would. I believe I could only have had this success due to the commitments I made to the group and this really made me strive to do the very best that I could. The result of all of this has been a deeper understanding of heuristic probability and the development of what I consider a really cool resource for teachers. It has been a long journey as well and at times I thought I may have taken on too much. My one regret is that I most likely will not get to work with such a diverse and great bunch of teachers again. Each of them brought something to the group, from one colleague's wacky and "out there" antics to another's steady and professional attitude. We were also very fortunate to have two very focused and helpful mentors who gave the project direction, commitment and rewards. Sadly we heard that funding has not been approved to continue this project on and I think, ultimately, someone has made a dumb decision there. Nevertheless, it has been a great ride and one that I have really enjoyed despite the occasional stressful moments.

Update: As it happens, there is one more twist to my adventure. I have been asked to give a workshop on heuristic probability at the Stats day at Auckland Uni. It is quite nice that I can reach some more teachers and show them what I did. I will certainly enjoy telling them of the learning process I went through. In some ways I feel I have gone from knowing nothing about this topic to being the local (secondary) expert on this.

Appendix F: Peter Radonich, Northcote College

Doodling and modelling

This report is based on the PowerPoint presentation I delivered at NZAMT to contribute to the section, “How I Came to My Topic”.

What got me interested.

I was initially inspired by a *New Zealand Herald* article about creating mathematical models of the heart by Professor Peter Hunter, a mathematician at the top of his field. I was also interested in the engineering component of the kinaesthetic art of another New Zealander at the top of his field, Len Lye.

The work of both involved struggle, but there were more connections. In a documentary on Len Lye, the artist discusses reading a newspaper article on a prominent young mathematician “Maths, it’s more like art”. Perhaps there was a similarity between the mathematical process that Peter Hunter goes through in developing his heart model and that of other creative processes such as drawing. Visual representation seemed important. After all, he made his heart *look* like a heart. I wondered about my students drawing and doodling within a mathematical environment such as graphical calculators or graphing software (or even something more dynamic). This led to a need to find out more about the process of developing a mathematical model, particularly in biology.

Modelling and drawing

My first contact was with Dr Piaras Kelly, whom I knew from my soccer team. I realised he worked in mathematical modelling from the teachers’ open day on calculus. He gave me DVDs on the heart and other models and I read some articles.

Mathematical models are often trying to behave and look like what they are modelling and this, certainly the latter part, is what kids are trying to do with their pictures. I therefore started to take greater note of my six-year-old son’s attempts at drawing aeroplanes and making models of them from cardboard.

I was thinking about Peter Hunter’s absorption with the heart model, and the doodling drawing process and the way students get absorbed when doodling or Oscar when drawing or making plane models. I wondered: Does this have anything to do with Csikszentmihalyi’s flow?

> The flow experience is when a person is completely involved in what he or she is doing, when the concentration is very high, when the person knows moment by moment what the next steps should be, like if you are playing tennis, you know where you want the ball to go, if you are playing a musical instrument you know what notes you want to play, every millisecond, almost. And you get feedback to what you're doing. That is, if you're playing music, you can hear whether what you are trying to do is coming out right or in tennis you see where the ball goes and so on. So there's concentration, clear goals, feedback, there is the feeling that what you can do is more or less in balance with what needs to be done, that is, challenges and skills are pretty much in balance. When these characteristics are present a person wants to do whatever made him or her feel like this, it becomes almost addictive and you're trying to repeat that feeling and that seems to explain why people are willing to do things for no good reason—there is no money, no recognition—just because this experience is so rewarding and that's the flow experience. (Mihaly Csikszentmihalyi)

Another quote that interested me:

> Math and Music can each work out similar niceties in finalising compositions but we're not all capable of developing our new brain intellect to solve mathematical problems—but all kids can get into art. (Len Lye)

After reading this I went to a lecture by Stephen Farthing. He suggests drawing should be part of curriculum in schools. Drawing includes estimating and measuring which sounds like it comes from the mathematics curriculum. "If you look at something and draw it, you'll begin to understand it" (Stephen Farthing).

Then I met James Sneyd, a mathematical physiologist. He said, "I look down the microscope and I'm trying to get the model on the computer screen to oscillate like the oscillations in the cell" and "I'm interested in the picture". He discussed how it is not easy to get the right shape and described telling a PhD student who complained of not being able to do it, "You've only tried once, try a couple of thousand of times."

Next I read about the Belousov-Zhabotinsky reaction, a spatio-temporal chemical oscillator. In 1951 the Russian scientist Boris Belousov discovered a solution that oscillated periodically between yellow and clear. He had discovered a chemical oscillator. The scientific community was united in believing this to be impossible. Some years later another Russian biophysicist, Anatol Zhabotinsky, refined the reaction, and discovered that when a thin layer of the solution is left undisturbed, geometric patterns such as concentric circles and Archemedian spirals propagate across the medium.

Another example: In 1963 the Polish mathematician Stanislaw Ulam was doodling in the interval between two seminars. He drew a grid of squares, then he wrote the number 1 at the centre of the grid and continued to write out the sequence of all the positive integers in ascending order spiralling out from the centre. Ulam noticed that when the integers were organised in this way, there was a tendency for the primes to be lined up along diagonal lines in the grid. The result was so unexpected that a picture of the Prime Number Spiral was featured on the cover of the March 1964 issue of *Scientific American*.

During class a student handed in a blank exam paper with a doodle on it. It was a conics paper and she had drawn a doodle in the Excellence section including a teddybear made of conics. I intend to get her to reproduce it using graphing software.

While thinking about introducing linear equations in class I came across the following quotes from Edward Laughbaum:

> When trying to apply the results of neuroscientific research to teaching algebra through the traditional equation-solving approach with traditional teaching methods (statement/examples/practice) we find that we are not maximizing the potential for learning. For example, starting a lesson with symbols, symbol manipulations, and logical deductive reasoning is not the best choice for promoting understanding or getting the attention of the students, especially if the content in the lesson is new.
>
> Working examples followed by practice exercises is not the best way to produce long-term memory or develop understanding. Teaching ‘applications’ after the algebra has been taught has little teaching value. Teaching algebra content as ‘stuff we will need later’ is not good for memory or attention.
>
> Memorizing algebraic processes through extended practice is only good in the short term—relative to memory and recall. Using a single method for teaching algebra more likely leads to false memories of the content taught.

I began, therefore, to find places to use the “flow” of doodling within my mathematics lessons. So with my students starting algebra I got the students to begin by creating and drawing families. Jake the Peg and his one-legged family is introduced first, extending to other families with other characteristics. The students draw families, then coordinates, then tables, then graphs, then form a formula and move into algebra last. Finally I gave the students a function and they had to produce the corresponding family.

Another example occurred when a colleague asked about ideas for teaching his Year 9 class linear equations, last period Friday. I suggested he try drawing pictures by plotting straight lines on graphics calculators. I observed the lesson and noted some student reactions: “I did it, look”, “I did it”, “Woooo … yeah”, “Look how beautiful that is Mister”. My colleague commented, “Normally I would have expected them to draw maybe four graphs, but most of them drew about 40.”

Now I am asking myself, “What next?” For example, for my own learning, I am asking myself how much the geometry of the HIV virus features in how it operates. For my teaching, I wonder what role 3D geometry software can play in my teaching.

Appendix G: Yoko Raike, Westlake Girls' High School

Modelling with differential equations

How I chose my topic

Initially I chose the topic "Rate" as a key concept which connects several ideas in mathematics. What I aimed to do was to make my teaching coherent throughout the year by maintaining connections among topics using "Rate" as a medium. Students see topics in mathematics, and standards in NCEA, as a disjointed set of knowledge and hence have difficulty making connections among them. In order to progress in mathematics to a higher level, the ability of being able to select and apply suitable skills to a problem from different areas in mathematics is essential. However, "Rate" is a broad area itself and I was unsure how I was going to fit this concept into the project. I asked for assistance from Judy. The following is taken from Judy's record of the meeting:

> Yoko wanted clarification about what she is expected to accomplish. I suggested that a possible cycle was examining some idea, getting more of an understanding of it, teaching some aspect, making different decisions about what to emphasise in her teaching … She was initially interested in ideas that connect concepts across the topics in senior secondary maths. … It was suggested that rate was a concept that could be 'seen' everywhere so she started to think about it.
>
> She spoke about a PD course she had been to where the presenter had shown examples of the use of differential equations. One involved the creation of electricity by electric eels. At the end of the talk the board was covered in DEs which meant very little to the teachers. A second talk looked at how maths modelling had helped clean up the Thames. This time she said the separate equations carried more meaning for her …
>
> She wants to find a situation that is interesting and relevant to the girls she teaches, and can be taught in a way that they can understand. … We thought biology was more likely to interest the girls than pure physics or chemistry. … She has been reading a book about CSI situations involving decaying bodies and wondered about using this scenario. Many girls watch CSI. … I am going to talk to James Sneyd and try to arrange for Yoko to meet him to ask him for suggestions; possibly he could talk to her students.

After the meeting I noted the following in order to organise my thoughts:

1. Area of mathematical knowledge: Mathematical modelling using differential equations and rates of change. The two projects (Recovering the Thames and Electric Eels) felt beyond my capability to understand fully.

2. How can I develop my knowledge on this topic? Studying models that have been developed by specialists and are in use. However, I do not think my mathematics is at the level where I can understand these models.

3. Use in my teaching. Motivating students.

I met with James Sneyd, a mathematical biologist, with Judy and another teacher in the project. The following is taken from Judy's record of the meeting:

> Yoko is looking for something the students can understand that is relevant and where the model represents the situation reasonably effectively. James suggested Alan Perelson's model of the rate of growth of the AIDS virus. $dx/dt = -kx$. James to send a preamble and discussion of it to others, he said the original paper in Nature is very dense. He talked about development of the model and what it shows about the behaviour of the virus which was contrary to what was believed by the clinicians.
>
> James suggested looking at excitable systems—spread of forest fires, the game of Life, Mexican waves and how their behaviour is determined by rules in same way P's pictures were determined by their functions. He talked about rules and games—Go and Othello. James to see if there are simple programmes that mimic spread of smallpox, forest fires.
>
> We started to think about a cycle of modelling in which you go from idea to doodle/fiddle to possible rule to picture and back round again as you try to get a better match between what you see and what the model gives you. We then talked about rules and how they determine games and the importance of playing in rule environments. Again the connection between rules—generating a picture on a computer—math equation—fixing the rules.
>
> Then P talked about a task/activity he had seen in which students shared a liquid which either changed or didn't change the other students' liquid as way of modelling spread. James talked about the spread of disease where some people are infected and some are immunised and what happens to the spread.
>
> The equation for this is $dx/dt = rx(1-x)(x-a)$ and the plot is a negative cubic going through 0, a and 1. Students can vary the values of a and r and see what happens to the epidemic. James will provide the equation for the spread of a virus. The function is difficult to determine and is expressed parametrically.

After the meeting I wrote down my intentions: to develop a clear understanding of the material; to make a lesson plan (when and how I should present this in class in order to have the maximum impact); and to use it in class.

I was aiming to achieve an increased level of motivation in my students, and hence an increasing level of understanding. I wanted to increase my own level of "fun" in teaching this topic, hence further motivating students.

How can the outcome be measured? I will not have any physical end products that can be presented as the result of my actions mentioned above. Test results will not reveal the level of the contribution of the action to the achievement of individual students unless I could conduct a designed experiment. I can describe my feelings but not students' feelings toward the action. Can Judy come to visit the lesson for evaluation?

Another matter is that the "AIDS virus" model is a one-off activity and I want more than one activity that could support my teaching and increase the motivation of students in calculus.

My basic concept was "Rate". The reason that it is important is that functions are another way of representing rate. Functions contribute a large part in mathematics, particularly at senior levels. Functions have applications in many areas and everyday life. This leads me to mathematical modelling. If my topic is mathematical modelling, it may be easier for me to find more examples of applications that can be shared in class.

In addition to the presentation of a simple model from a real study (AIDS), I'd like to try fitting models to real-life situations. I would start with a very basic function, which may be ridiculous in the eyes of specialists, but not to Year 12 students. Then I would find another model with a slightly harder function and so on. After discussing the goodness of fit of a model with students, I would eventually introduce a final model. The final model would be more complex than I could handle but this does not matter. If I am successful in doing this, I am sure some students will see it as an interesting challenge in their future. I think one of the biggest difficulties we face in class is that the knowledge used in real problems, and that we are trying to convey to students, is too huge even for senior students. There are some attempts in textbooks to add real-life examples. However, I feel the real-life context is not exciting enough to raise the curiosity of students since it stops at a level lower than that at which students can understand, and hence trivialises the topic. Teachers leading students through trial and error processes, showing the progression of a model and acknowledging that some of the models are currently beyond their ability would assist in giving students an opportunity to free themselves from the common image of mathematics that there is one correct solution derived from one perfect method. Studying simpler functions can then be seen as building up knowledge to get there eventually.

I have been looking for something I can start modelling, but so far without success. I have one short section of a TV programme which I will use in class to introduce "Rate" and "Modelling".

I think students may ask why it is necessary to take the approach of using a differential equation to come up with a model instead of simply collecting data and coming up with a model directly from that. For example, in order to model how fast a cup of coffee cools down we can collect data from experiments and average the results to come up with a model (function). Why must we start from the rate of change and integrate it into a model? I do not think I can answer this question effectively.

My learning process

What James Sneyd provided us is great. The story of the AIDS research is intriguing, and the model is simple enough that Year 13 students will be able to understand it.

I looked through his book for more examples. It soon became apparent that the book was too advanced for me, yet I am very happy with the information that I gained from the first few pages. Seeing me struggling with it, James lent me another book, which I assume is stage one level reading material. This book answered my fundamental question of why are differential equations so important? The role of mathematical models is explained in the second page of this book. We, or maybe I, tend to focus on correct answers and correct methods when dealing with mathematical problems. A common belief, particularly among students, is that the outcome must be "correct". We tend to devalue things that are incorrect. However, both the success and the failure of differential equation models formed around possible explanations to natural phenomena and contribute to a further investigation process. Differential equations are thus a tool that probes natural phenomena.

After some reading, I started to create a presentation for my calculus class using the HIV virus model. While creating the presentation, I thought the model was "fun", and I wanted to show it at the start of the differential equations section in order to motivate students. On the other hand, if I used this at the start, it was likely to be wasted as students do not have enough background knowledge to appreciate it. By the time I finished the first draft, I also wanted to give this opportunity to my Year 12 extension students. Hence, I modified the presentation to suit both year levels. This led me to another thought. Those Year 13 students in my class and who are taking both calculus and statistics may notice that if they plot observed data in Excel, they will get a best fit model and its equation, and could wonder that the point of learning differential equations is. Learning differential equations looks harder than learning Excel. At this introductory stage of differential equations, I concluded that I should focus on explaining why using differential equations is preferable to simply fitting curves to data in order to establish models. Hence, for the introduction, I used Newton's Law of Cooling, a topic to which students could relate to more easily.

Learning as a group

It was interesting to notice that another teacher in the project and I shared the same topic and were provided with the same material, but how we used them and what we saw in them were very different.

I worked individually. The level of collaboration between other members in the group and myself may not have reached the level the project may have had expected to see. However, even if this is true it does not devalue having the project as a group. What has been generated through the group was the energy for carrying out the project. Judy's contribution was huge but if I were working alone with her, I do not think I would have produced the outcome at the same level as I did with the group. It is a basic human psychology that gains the sense of security from a group. At the start I was not certain whether I could produce an outcome that was expected from the project. This uncertainty and concern was eased by the sense of belonging to a group of people who were aiming to do the same thing.

Impact on students

Starting with Newton's Law of Cooling with my Year 13 class was a jump compared to using a simple example. However, I do not think that having started the differential equations section with Newton's Law of Cooling made it harder for my students. I feel that the introductory lesson went more smoothly than ones before. Starting with a familiar situation must have helped taking pressure off the students who are meeting a new concept. I think that presenting the reason for using differential equations has made students more open to this new idea. I did not have any students saying, "Why do I need to learn this?" I showed the HIV virus presentation to interested Year 12 and Year 13 students as a lunchtime open session, which attracted the interest of many. The following extract from a message to James Sneyd shows how I felt after the lunchtime presentation:

> Dear Dr. Sneyd
>
> The attached PowerPoint presentation, which I created from your material, is what I did with our students during lunchtime. I modified and structured your material to suit my style of teaching and to the level that I think our students can follow. My first plan was to use it in my calculus class but it ended up being used for this lunchtime session for both Yr 12 and Yr 13 students, and was a great success. At least 40 students filled up my classroom. Interestingly, almost all were Yr 12. I was wondering how much of the presentation made sense to them, which was answered next day by a student walking in my classroom saying, 'Mrs. Raike it was awesome yesterday!' All of the university brochures were taken. The same week, I ran a lunchtime tutorial focused on Excellence level Algebra for Yr 12, which also had an impressive turnout. In the past, only a handful of students showed. This year I opened the tutorial to two Cambridge extension classes and this may have contributed to the increased attendance. However, I am sure that I saw more than a handful of unfamiliar faces this time. Interest is there. The question is how to support and maintain the interest of students in our subject.

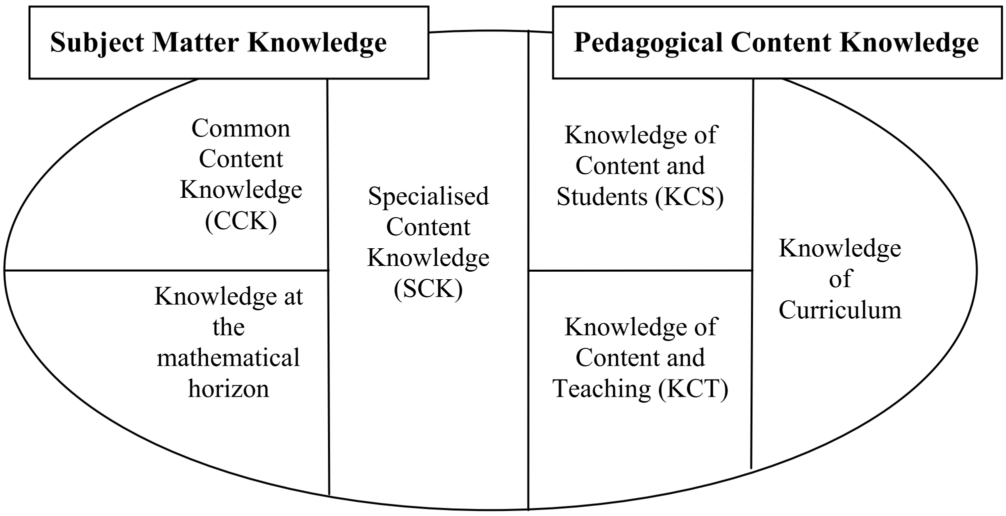
Reflection on the project

Reflecting back on what I wrote and what I did, I realise that at the end this project produced an answer to my fundamental question of why differential equations are useful, a question I did not originally regard as the fundamental. My primary desire was to learn how to solve or understand those complicated differential equations used in real-life projects. My assumption was that if I could understand differential equations more by acquiring mathematical skills, the understanding of why differential equations are useful would follow naturally and I would be happier teaching this topic in class. Hence the question of why are differential equations useful was thought to be of secondary importance. However, it turned out that the passage in *Population Biology* **(Ref.)** gave me great joy in teaching this topic in spite of totally no progress in my skills in solving differential equations. This does not mean that all the other things I experienced or learnt throughout the project are not important. Without those discussions, thoughts and the exciting example of the AIDS virus research, the meaning of the passage would not have had as much impact on me.

I wondered why I did not develop this understanding of the usefulness of differential equations when I was a learner. My conclusion is that, as a learner, I focused only on how to solve equations. The rewards of learning at that time came from success in examinations. The comment my daughter made when she was proofreading my PowerPoint slides was, “Only dedicated students will read this.” This agrees with my observation above and raises further questions. Does the statement I made earlier, “the passage in *Population Biology* gave me great joy in teaching this topic”, have any value to learners? Is it worth explaining the usefulness of differential equations is class? Is the current system of teaching differential equations and log modelling in two separate courses, calculus and statistics, helping learners build a coherent understanding of mathematics? I believe that it is worth explaining the usefulness of differential equations in class and a teacher having “fun” with the topic must have an impact on his/her teaching. I do not believe that the separation of the topics is beneficial to students’ learning.

Appendix H: Ball/Bass model of mathematical knowledge for teaching

Shulman's original category scheme (1985)
compared with the Michigan model



Source: Delaney, Ball, Hill, Schilling, & Zopf, 2008, p. 179.